Midterm MATH 251, Fall 2015

Justify all answers

Problem 1 [5 pt] Let V be the vector space indicated below, case by case. In each case find if the specified vector $\mathbf{v}$ belongs to the span of the given set S.

$$V = \mathbb{R}^4; \quad \mathbf{v} = (1,2,0,1); \quad S := \{(1,0,1,1),(1,1,1,1),(2,1,2,2),(0,1,0,0),(0,0,0,1)\} \tag{1}$$
$$V = \mathcal{P}_4(\mathbb{R})\ , \quad \mathbf{v} = 3x^3 + 2x - 1, \quad S := \left\{x^4+2, 2x-1, x^3, x^3+3\right\} \tag{2}$$

Problem 2 [5pt] We denote by $\mathcal{C}(\mathbb{R})$ the vector space of continuous functions on $\mathbb{R}$. Show whether the following three vectors are linearly independent or not

$$V := \mathcal{C}(\mathbb{R}); \quad f_1(x) = \mathrm{e}^x; \quad f_2(x) = \mathrm{e}^{3x}; \quad f_3(x) = x(x-1) \tag{3}$$

Problem 3 [5 pt] Let $V = Mat_{2\times 3}(\mathbb{R})$ and consider the following subspaces:

$$W_1 := \left\{ \begin{bmatrix} a & b & a+b \\ c+d & d & d \end{bmatrix} \ a,b,c,d \in \mathbb{R} \right\}, \qquad W_2 := \left\{ \begin{bmatrix} f & 2f & g \\ e & e & \ell - e \end{bmatrix} \ e,f,g,\ell \in \mathbb{R} \right\} \tag{4}$$

Find bases and dimensions of $W_1,\ W_2,\ W_1+W_2,\ W_1\cap W_2$.

Problem 4 [5 pt] Let $T : V \to W$ be a linear transformation between two finite dimensional vector spaces. Suppose that $\dim V = 2 + (\dim W)$. Is it possible for T to be one-to-one? If yes give an example, if not explain why.

Problem 5 [5 pt] Let $T : \mathcal{P}_2 \to \mathbb{R}^2$ be given by $T(p(x)) = \langle p(3), p'(2)\rangle$. Let $\beta = \{x+2, x-3, x^2\}$ be a basis of $\mathcal{P}_2$ and $\alpha = \{1, x, x^2\}$ another basis. Let $\gamma = (\langle 1,1\rangle, \langle 1,-1\rangle)$ be a basis of $\mathbb{R}^2$. Compute $[T]_\beta^\gamma$, $\quad [T]_\alpha^\gamma$.

Problem 6 [5 pt]

Consider the transformation $T : \mathcal{P}_2 \to \mathcal{P}_4$ where $\mathcal{P}_n$ denotes the finite dimensional vector space consisting of polynomials of degree up to n:

$$T(p(x)) = p(x^2) + p(x-1) \tag{5}$$

Note: here $p(x-1)$ means the shift of variable, for example if $p(x) = x^2+2$ then $p(x-1) = (x-1)^2+2 = x^2-2x+3$; similarly $p(x^2)$ means the change of variable, for example if $p(x) = 2x^2+x$ then $p(x^2) = 2(x^2)^2 + (x^2) = 2x^4 + x^2$.

1. Show that T is linear;
2. Find $[T]_\beta^\gamma$ where $\beta = (1, x, x^2), \gamma = (1, x, x^2, x^3, x^4)$ are the standard ordered bases of $\mathcal{P}_2, \mathcal{P}_4$, respectively.

Problem 7 [Bonus 3 pt] Let $T : V \to W$ and $U : V \to W$ be two linear transformations between the indicated vector spaces. Prove that $\mathbf{R}(U+T) \subseteq \mathbf{R}(U) + \mathbf{R}(T)$, where $\mathbf{R}(U),\ \mathbf{R}(T)$ denote the *ranges* of the indicated transformation. Give an example where the inclusion is strict.